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The drift velocity of water waves

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The important role of viscosity in producing second-order Eulerian drift currents in the presence of small-amplitude water waves was first recognized by Longuet-Higgins (1953).

The theoretical and experimental background is first reviewed. It is then shown that, contrary to previous belief, the presence of surface contamination must greatly enhance the drift velocity of short waves. We then solve an initial-value problem for the drift current associated with temporally decaying waves, thereby resolving questions raised by the work of Liu & Davis (1977), whose solution exhibits anomalous singularities. Next, the steady drift velocity of spatially decaying waves is calculated and shown to bear a close resemblance to Longuet-Higgins' 'conduction solution' for unattenuated waves.

Finally, we establish that unidirectional drift currents of both surface and interfacial waves are sure to be unstable to spanwise-periodic disturbances; the instability mechanism being identical to that first proposed by Craik (1977), and recently developed by Leibovich & Paolucci (1981), to explain the generation of Langmuir circulations.

1. Introduction

Small-amplitude progressive surface gravity (or capillary-gravity) waves are known to induce mean drift velocities in the direction of wave propagation. Stokes (1847) first predicted the drift velocity using inviscid theory, but Longuet-Higgins (1953) established that the role of viscosity remains important even as the characteristic Reynolds number R approaches infinity. Much earlier, Rayleigh (1896, p. 340) had reached a similar conclusion for a related problem.

More precisely, Longuet-Higgins showed that, just outside the thin oscillatory viscous boundary layer near the channel bottom, there is a constant $O(a^2)$ mean drift velocity, where a is a measure of wave amplitude; and also that, just beyond the corresponding boundary layer near the free and *uncontaminated* surface, there is a mean $O(a^2)$ gradient of drift velocity. Both these quantities persist even as $R \to \infty$ and may be regarded as boundary conditions for the mean Eulerian velocity $\bar{u}_e(z)$ in the interior of the liquid. Whereas $\bar{u}_e(z)$ is identically zero according to purely inviscid theory, these results prove that this is not the case for real liquids. The mean Lagrangian drift velocity, or mass-transport velocity, is the sum of $\bar{u}_e(z)$ and Stokes's drift profile $\bar{u}_s(z)$.

For waves in closed channels, it is appropriate to impose a condition of zero net mass flux at each downstream location x in calculating the steady-state drift velocity.

This requires that a horizontal hydrostatic pressure gradient be established by 'set up' of the free surface: that is, the mean depth must increase linearly with x. For spatially periodic waves in an unbounded domain, there is no such pressure gradient and the steady-state mass flux may be non-zero.

Various experiments confirm the existence of mean drift-velocity profiles (see § 3 and Liu & Davis 1977) and certain features of the theoretical solution are in reasonable quantitative agreement with *some* of the experimental data. But it is fair to say that convincing agreement between theoretical and experimental profiles over the entire liquid depth is lacking. The work of the present paper indicates several possible reasons for this discrepancy, which are additional to those advanced by Dore (1977, 1978). Particular attention is given to the effect of surface contamination and to a correction and extension of the work of Liu & Davis (1977) on temporally decaying waves. The case of spatially decaying waves is also examined.

These extensions of the conventional theory, and the other extensions by Dore (1977, 1978) and Grimshaw (1981), are still unlikely, however, to provide good general agreement with experiment. Rather, it is argued that these theoretical drift profiles are usually *inherently unstable* to disturbances periodic in the spanwise direction. The mechanism for this instability is identical to that proposed by Craik (1977) and developed by Leibovich & Paolucci (1980, 1981) for the generation of Langmuir circulations; it differs from this previous work only in that the mean Eulerian current $\bar{u}_e(z,t)$ here derives purely from the wave motion rather than from an applied wind stress at the surface. The drift velocity of interfacial waves is also briefly considered and similar conclusions are drawn regarding its instability to spanwise-periodic perturbations.

2. The theoretical background

2.1. *Periodic waves*

An inviscid irrotational surface wave with upwards surface displacement

$$z = a\cos(kx-\sigma t)$$

and small amplitude a has an associated velocity potential

$$\phi(x,z,t) = \frac{\sigma a\cosh[k(z+d)]}{k\sinh kd}\sin(kx-\sigma t), \tag{2.1}$$

correct to $O(a)$, where the Eulerian velocity is $\mathbf{u} = (u,w) = \nabla\phi$, the mean water depth is d and z is measured vertically upwards from the mean surface $z = 0$. The frequency σ satisfies the linear dispersion relation $\sigma^2 = [gk+\rho^{-1}\gamma k^3]\tanh kd$, where g is gravitational acceleration, ρ the liquid density and γ the coefficient of surface tension. For such spatially periodic waves, there is no $O(a^2)$ mean Eulerian current according to inviscid theory but Stokes (1847) found the mean Lagrangian, or mass-transport, velocity to be

$$\bar{u}_s(z) = \frac{\sigma ka^2\cosh[2k(z+d)]}{2\sinh^2 kd} \tag{2.2}$$

(see for example Phillips 1977, chap. 3).

Longuet-Higgins' (1953) analysis of viscous effects is briefly summarized by Phillips (1977, pp. 54–58). Longuet-Higgins supposed that the waves are purely periodic in x and t and that the mean-drift profiles are functions of z alone. But, to maintain a purely periodic wave in viscous liquid, it is necessary to do external work on the liquid and this can only be accomplished by the application of suitable stresses at the surface. Since Longuet-Higgins' analysis requires that tangential stresses at the surface are zero, it must be supposed that periodic normal stresses are applied of such magnitude and phase as to maintain the wave at constant amplitude, despite viscous dissipation.

With an uncontaminated surface, most of the energy dissipation takes place within the bottom boundary layer if

$$\beta/k \gg \sinh^2 kd, \quad \beta \equiv (\sigma/2\nu)^{\frac{1}{2}}.$$

This bottom boundary layer has thickness $O(\beta^{-1})$, and ν denotes the kinematic viscosity of the liquid. The normal stresses then support a periodic pressure distribution in the interior which does work on the bottom boundary layer at a rate which exactly balances the rate of energy dissipation by viscosity within it. This pressure distribution has a slight phase shift relative to that given by inviscid theory.

A steady $O(a^2)$ mean Eulerian velocity $\bar{u}_e(z)$ satisfies

$$\nu\frac{d^2\bar{u}_e}{dz^2} = \frac{d}{dz}(\overline{uw}) + \frac{1}{\rho}\overline{\frac{dp}{dx}}, \tag{2.3}$$

where $\overline{dp/dx}$ denotes a constant imposed horizontal pressure gradient (which may either be set equal to zero for unbounded channels or chosen to yield zero total mass flux for closed channels), and $-\rho\overline{uw}$ denotes the time-averaged Reynolds stress τ_{xz} resulting from the wave field.

For the irrotational wave field (2.1), $\overline{uw}$ is identically zero. But treatment of the viscous boundary layer adjacent to the bottom, which has thickness of order

$$\beta^{-1} \equiv (2\nu/\sigma)^{\frac{1}{2}} \ll d, k^{-1}$$

leads to

$$\overline{uw} = \frac{\sigma^2 a^2 k}{4\beta\sinh^2 kd}\{2e^{-\beta z'}(\beta z'\sin\beta z' + \cos\beta z') - 1 - e^{-2\beta z'}\}, \tag{2.4}$$

where $z' \equiv z+d$ (cf. Phillips 1977, equation (3.4.30)). Outside the bottom boundary layer, $\beta z' \to \infty$ and $\overline{uw} \to (\overline{uw})_\infty$, a constant given by

$$(\overline{uw})_\infty = \frac{-\sigma^2 a^2 k}{4\beta\sinh^2 kd}. \tag{2.5}$$

The Reynolds stress remains constant throughout the region where the wave motion is irrotational. It is non-zero because the velocity potential has a correction $O(\sigma a\beta^{-1})$, induced by the bottom boundary layer, which is out of phase with (2.1). This correction is

$$\phi_1 = \frac{\sigma a\cosh kz}{2\beta\sinh^2 kd}[\cos(kx-\sigma t) + \sin(kx-\sigma t)],$$

which results in the non-zero Reynolds stress $-\rho(\overline{uw})_\infty$ given in (2.5). For infinite depths $kd \to \infty$, both ϕ_1 and the Reynolds stress $(\overline{uw})_\infty$ are zero.

If it is assumed that $d\overline{u}_e/dz \to 0$ as $\beta z' \to \infty$, if $\overline{dp/dx}$ is meantime taken as zero, and if the no-slip condition $\overline{u}_e(-d) = 0$ is applied, integration of (2.3) leads to

$$\overline{u}_e = \frac{\sigma a^2 k}{4\sinh^2 kd}[3-2(\beta z'+2)\,e^{-\beta z'}\cos\beta z' - 2(\beta z'-1)\,e^{-\beta z'}\sin\beta z' + e^{-2\beta z'}] \tag{2.6}$$

(cf. Phillips 1977, equation (3.4.33)). This describes the mean flow within the bottom boundary layer, and it is readily seen that

$$\overline{u}_e = \tfrac{3}{4}\sigma a^2 k \operatorname{cosech}^2 kd \tag{2.7}$$

just outside this boundary layer (i.e. at distances $z' = \Delta$ such that $\beta^{-1} \ll \Delta \ll k^{-1}$: we dispense with the formality of the multiple-scales technique in this paper, though the idea remains implicit). Result (2.7) added to the Stokes drift $\overline{u}_s(-d)$ yields a 'bottom' mass-transport velocity which is $\frac{5}{2}$ times the inviscid result, or

$$(\overline{u}_e+\overline{u}_s)_{z=-d+\Delta} = \tfrac{5}{4}\sigma a^2 k \operatorname{cosech}^2 kd. \tag{2.8}$$

Near the liquid surface, there is also a thin oscillatory boundary layer which is best studied using curvilinear co-ordinates which fit the wavy surface. With a clean surface, this viscous boundary layer is rather weak, having vorticity $O(ak\sigma)$; but Longuet-Higgins showed that it is nevertheless responsible for inducing a mean second-order vorticity outside this boundary layer such that

$$(d\overline{u}_e/dz)_{z=-\Delta} = 2\sigma a^2 k^2 \coth kd. \tag{2.9}$$

This gives a vertical gradient of mass transport

$$[d(\overline{u}_e+\overline{u}_s)/dz]_{z=-\Delta} = 4\sigma a^2 k^2 \coth kd, \tag{2.10}$$

or twice the value predicted by Stokes.

Conditions (2.7) and (2.9) may be regarded as boundary conditions which, together with (2.3), determine the mean Eulerian flow in the interior of the liquid, outside the oscillatory boundary layers on either wall. It is readily confirmed that the omission of $\overline{dp/dx}$ in deriving (2.7) incurs an error $O(k\Delta)$, which is negligible. Likewise, these boundary conditions may be imposed at $z = -d$ and $z = 0$, setting $\Delta = 0$ with negligible error. Of more concern is the assumption that $d\overline{u}_e/dz \to 0$ outside the bottom boundary layer, which was used in deriving (2.6). This assumption is apparently inconsistent with (2.3); for, in the interior, $\partial(\overline{uw})/\partial z$ is zero, and

$$d\overline{u}_e/dz = (\nu\rho)^{-1}\,(\overline{dp/dx})\,z + 2\sigma a^2 k^2 \coth kd, \tag{2.11}$$

on using (2.9). Hence the correct matching condition for the bottom boundary layer as $\beta z' \to \infty$ is

$$d\overline{u}_e/dz \to -(\nu\rho)^{-1}\,(\overline{dp/dx})\,d + 2\sigma a^2 k^2 \coth kd. \tag{2.12}$$

This may indeed be ignored in the leading-order calculation which yields (2.7), since it gives only a correction $O(k\Delta)$ in $\overline{u}_e$. However, this $O(a^2)$ velocity gradient is always present and so introduces some uncertainty in the interpretation of velocity measurements 'just outside the boundary layer'. A more meaningful quantity is the local maximum of $\overline{u}_e$ which occurs near the bottom (see Longuet-Higgins 1953, figure 4).

Integrating (2.11) and using condition (2.7) leads to the Eulerian velocity distribution

$$\overline{u}_e(z) = \tfrac{1}{2}(\nu\rho)^{-1}\,(\overline{dp/dx})\,(z^2-d^2) + 2\sigma a^2k^2(z+d)\coth kd + \tfrac{3}{4}\sigma a^2k\,\mathrm{cosech}^2\,kd. \quad (2.13)$$

For unbounded channels, $\overline{dp/dx} = 0$; while, for closed channels, the zero-mass-flux condition

$$\int_{-d}^{0} (\overline{u}_e + \overline{u}_s)\,dz = 0$$

yields

$$\overline{dp/dx} = 3d^{-3}\nu\rho[\sigma a^2(\tfrac{1}{2}+k^2d^2)\coth kd + \tfrac{3}{4}\sigma a^2kd\,\mathrm{cosech}^2\,kd]. \quad (2.14)$$

In the latter case, the mass-transport velocity is

$$\overline{u}_e + \overline{u}_s = \frac{\sigma a^2\coth kd}{d^2}(z+d)\left[\frac{3}{4}\left(\frac{z}{d}-1\right) + \tfrac{1}{2}k^2d^2\left(\frac{3z}{d}+1\right)\right] + \frac{\sigma a^2k}{2\sinh^2 kd}\left\{\frac{9}{4}\left(\frac{z}{d}\right)^2 - \tfrac{3}{4} + \cosh[2k(z+d)]\right\}, \quad (2.15)$$

which is just Longuet-Higgins' (1953) 'conduction solution' for a progressive wave (see his equation (300)). Unlüata & Mei (1970) give an alternative derivation of this solution, using Lagrangian dynamical equations from the outset. Dore (1970, 1978*a*, *b*) has calculated the corresponding mass-transport velocities of interfacial waves in a two-fluid system (see § 7).

Longuet-Higgins also presents a 'convection solution' for the mean flow, arguing that, under typical conditions, additional nonlinear convective terms in the mean-flow equations are likely to outweigh viscous diffusion. This is so for standing waves. It is also true for some considerable distance downstream of a wavemaker where the viscous boundary layers grow progressively in thickness (see Dore 1977) and may be true for mean flows associated with modulated packets of progressive waves (see Grimshaw 1981). However, for the unidirectional and unbounded mean flows envisaged here, the convective terms are identically zero. In this case, (2.15) is the appropriate steady solution in two dimensions, attained after a time (or distance from a wave maker) sufficiently long for vorticity to diffuse throughout the whole fluid; and no restriction to small mean-flow Reynolds numbers max $|\overline{u}_e|\,d/\nu$ is necessary. But, if the depth d is infinite, this steady state is never attained: a 'triple-deck' model with ever-deepening boundary layer is then appropriate (Dore 1977; Grimshaw 1981) until the penetration depth becomes sufficiently large that Coriolis effects are significant. A steady-state solution for infinite depth with Coriolis force has been given by Madsen (1978).

2.2. *Decaying waves*

If the liquid surface is truly free, with no applied normal stresses, the wave amplitude decays exponentially by viscous action at a temporal rate

$$\sigma_i = \tfrac{1}{2}\sigma\beta^{-1}k\,\mathrm{cosech}\,2kd \quad (2.16)$$

(cf. Phillips, equation (3.4.29)), provided the liquid depth is sufficiently small that $\beta/k \gg \sinh^2 kd$. Alternatively, for deep-water waves with $\beta/k \ll \sinh^2 kd$, the damping rate is $\sigma_i = 2\nu k^2$, which results from the viscous stresses within the irrotational motion.

For intermediate depths a satisfactory approximation is given by the sum of these two expressions for σ_i:

$$\sigma_i = \tfrac{1}{2}\sigma\beta^{-1}k \operatorname{cosech} 2kd + 2\nu k^2. \tag{2.17}$$

Waves may decay spatially with x, rather than temporally with t; to good approximation, this spatial attenuation rate is $\sigma_i(d\sigma/dk)^{-1}$ when σ_i is small. These results may be drastically modified by the presence of surface contamination or the influence of the viscosity of air.

For temporally decaying waves, the results of § 2.1 for the drift velocity are no longer strictly applicable. In particular, it is no longer meaningful to talk of a steady-state mass-transport velocity, reached at suitably long times after the onset of wave motion. The influence of temporal decay has been investigated by Liu & Davis (1977). To linear approximation, the wave field is just as for a constant-amplitude wave, apart from the inclusion of the decay factor $\exp(-\sigma_i t)$. Accordingly, the mean Reynolds stress $-\rho\overline{uw}$ is just as before, but with an additional factor $\exp(-2\sigma_i t)$, and the boundary conditions for the second-order mean flow are

$$(d\bar{u}_e/dz)_{z=0} = 2\sigma a^2k^2 \coth kd\, e^{-2\sigma_i t}, \tag{2.18}$$

$$(\bar{u}_e)_{z=-d} = \tfrac{3}{4}\sigma a^2 k \operatorname{cosech}^2 kd\, e^{-2\sigma_i t}, \tag{2.19}$$

analogous to (2.7) and (2.9) (see Liu & Davis, equations (4.5*b*) and (5.10)).

In an Eulerian description, all the wave momentum is contained in the region above the wave troughs (see Phillips 1977, p. 40). The loss of this momentum, no longer replenished by surface forces, supports both the Reynolds stress $-\rho(\overline{uw})$ and the viscous stress $\rho\nu(d\bar{u}_e/dz)$ at $z = 0$.

The mean x-momentum equation outside of the viscous boundary layers (taking averages with respect to x, not t) is

$$\frac{\partial \bar{u}_e}{\partial t} - \nu\frac{\partial^2 \bar{u}_e}{\partial z^2} = -\frac{1}{\rho}\overline{\frac{\partial p}{\partial x}}, \tag{2.20}$$

since $\overline{uw}$ is constant in this region, where now $\bar{u}_e$ is a function of both z and t, and the pressure gradient $\overline{\partial p/\partial x}$ is a function of t only.

Liu & Davis give a particular solution of (2.20), which satisfies (2.18) and (2.19). This corresponds to the decay rates σ_i of deep-water waves with $\beta/k \ll \sinh^2 kd$, namely

$$\sigma_i = 2\nu k^2 = \sigma k^2/\beta^2, \tag{2.21}$$

rather than (2.16), though they supposed the depth to be finite. This inconsistency was criticized by Dore (1978*b*) and the derivation of an equivalent solution for the decay rate (2.16) was given by Knight (1977). More generally, a solution of (2.18) to (2.20), valid for *any* small decay rate σ_i given by (2.17), is

$$\bar{u}_e(z,t) = \frac{1}{\rho}\overline{\frac{\partial p}{\partial x}}\frac{1}{2\sigma_i}\left(1-\frac{\cos\delta z}{\cos\delta d}\right) + e^{-2\sigma_i t}\left\{2\sigma a^2k^2\coth kd\left(\frac{\sin[\delta(z+d)]}{\delta\cos\delta d}\right)\right.$$
$$\left. + \tfrac{3}{4}\sigma a^2 k \operatorname{cosech}^2 kd\left(\frac{\cos\delta z}{\cos\delta d}\right)\right\}, \tag{2.22}$$

where $\delta \equiv (2\sigma_i/\nu)^{\frac{1}{2}}$ and $\overline{\partial p/\partial x} = P e^{-2\sigma_i t}$ (P constant).

This solution is singular at depths such that $\cos \delta d$ vanishes. At depths other than these, the constant P may be chosen to yield zero total mass flux at each instant t. This solution is clearly rather different from the steady conduction solution (2.12). Liu & Davis argue that, at the depths $\delta d = \frac{1}{2}n\pi$ ($n = 1, 3, 5, \ldots$) for which the solution (2.22) is singular, no quasi-steady state varying as $\exp(-2\sigma_i t)$ can result from an initial-value problem with $\bar{u}_e(z, 0)$ initially zero.

Of course, these singularities are physically irrelevant, since diffusion problems with finite initial and boundary values have only smooth solutions. This is confirmed in § 5, where the initial-value problem is solved.

For spatially decaying waves, the temporally averaged second-order Eulerian velocity $\bar{\mathbf{u}}_e$ necessarily has a vertical as well as a horizontal component and so the convective terms $(\bar{\mathbf{u}}_e . \nabla)\bar{\mathbf{u}}_e$ may not be negligible. This situation is examined in § 6.

2.3. *Surface contamination*

When the liquid surface is contaminated by a surface-active agent such as a thin – perhaps monomolecular – layer of detergent or certain oils, the character of the viscous boundary layer near the surface is much altered. Now, the surface can sustain tangential stresses due to changes in surface tension associated with varying surface concentrations of contaminant. The surface acts essentially as an elastic membrane which resists extension and contraction, though other physical effects such as surface viscosity and transfer of soluble contaminant between the surface and the bulk liquid may sometimes be significant. For a discussion of such matters, and further references, see Miles (1967), Smith & Craik (1971) or Gottifredi & Jameson (1968).

While most of the energy dissipation with a clean surface occurs in the bottom boundary layer or, for deep-water waves, in the irrotational wave field, contamination enhances the dissipation in the boundary layer near the surface to such an extent that it may become the largest contribution. The influence of such contamination is modelled most simply by supposing that the surface is rendered inextensible though free to move as a whole: that is to say, the surface elasticity parameter is taken to be very large. This model yields an attenuation rate of

$$\sigma_i = \tfrac{1}{4}\sigma\beta^{-1}k \coth kd \tag{2.23}$$

deriving from this surface layer alone (cf. Phillips, equation (3.4.28)), which exceeds (2.16) and (2.21). Although genuinely inextensible surface films are not encountered in nature (see Gottifredi & Jameson 1968), this 'inextensible limit' is known to yield results for short waves which are in broad agreement with those of real films with even moderately small elasticity. Indeed, the maximum damping rate of short waves occurs at such a moderate value, and is just twice that shown in (2.23). However, as the wavelength increases, the influence of real surface films decreases: a fact not reflected by the inextensible model. In practice, short capillary-gravity waves are those most affected by contamination and for which the inextensible model is an acceptable approximation.

For such waves, it is plausible that contamination may modify the second-order drift velocity. However, Phillips (1977, p. 58) suggests that the presence of contamination does *not* alter the mean vorticity (2.9) beneath the surface boundary layer, but simply induces a net velocity change across the layer (which can have no influence upon the drift velocity in the interior region. A similar conclusion was reached by

Huang (1970) and Dore (1972), but Huang employed an incorrect boundary condition and Dore does not give full analytical details.

In fact, their conclusion is certainly wrong. Phillips argues that the enhanced momentum loss due to contamination is accommodated by a non-zero Reynolds stress, even in deep water. This is an untenable view, however, since $\overline{uw}$ is constant in inviscid regions and $u, w \to 0$ as $z \to -\infty$. Instead, the increased momentum loss-rate must induce a correspondingly greater mean vorticity just beneath the surface. The only case where this does not occur is that of a surface film which supports an appropriate *mean* $O(a^2)$ tangential stress as well as fluctuating $O(a)$ stresses. But such a mean stress necessitates a continuous, linear change of surface tension with downstream distance x, brought about by a corresponding change in concentration of contaminant: a situation likely to obtain only over rather short distances near the 'edges' of a slick. It is shown in § 4 that, when the mean concentration of contaminant is independent of x, the boundary condition (2.9) *is* substantially modified by contamination, with consequent large changes in the theoretical drift velocity.

3. The experimental evidence

Evidence in support of the predicted drift velocity (2.8) near the channel bottom is fairly strong. Collins's (1963) results are the most comprehensive (and are reproduced in Phillips 1977, p. 56) but those of Russell & Osorio (1957) show similar agreement. Such good agreement is perhaps surprising, since the bottom boundary layer was observed to be unstable, or turbulent, in many of the reported cases: Longuet-Higgins, in an appendix to Russell & Osorio's paper, hazards an explanation of this. Even when the bottom boundary layer is turbulent, which according to Collins occurs when

$$(\sigma/2\nu)^{\frac{1}{2}}\, a \operatorname{cosech} kd \gtrsim 80,$$

this turbulence does not spread far from the wall: Russell & Osorio report typical depths of about 1 inch for this turbulent region.

Longuet-Higgins (1960) himself made experimental observations to test result (2.10) for the gradient of mass transport near the surface. These visual studies of a deforming dye streak revealed a gradient quite close to (2.10) and certainly much greater than Stokes' prediction. In contrast, Russell & Osorio's observations show variable gradients which are typically larger than (2.10). One difference between the experimental conditions for these two sets of observations (made, incidentally, in the same channel) should be noted. Whereas Longuet-Higgins attempted to ensure a clean surface, Russell & Osorio deliberately introduced Teepol, a strongly surface-active detergent, 'in order to disperse the film of dirt which invariably formed' and to obtain 'surface drift velocities... stable and uniform across the section'. The difference in their results suggests that surface contamination may influence the drift-current gradient below the surface boundary layer. Furthermore, Dore (1978*a*) has pointed out that good agreement with result (2.10) is not to be expected, even with a clean surface, owing to enhanced dissipation associated with viscous effects in the air above the free surface.

Drift-velocity measurements which cover the whole depth of water are given by Russell & Osorio (1957), who determined the velocities of neutrally buoyant particles at various depths. They report that it 'was possible to obtain drift velocities... independent of (downstream and spanwise) position and time. It is probable, however, that the velocities would be distorted by circulations in a horizontal plane if the waves were not confined to a narrow channel.' It was also pointed out by Longuet-Higgins (1953, p. 577) that 'it is by no means certain that a steady state will exist which is compatible with the boundary conditions at both the wavemaker and at the wave absorber or that, if it exists, it is stable.' The question of stability of the drift profile is examined in § 7.

Russell & Osorio's profiles bear, at best, only a qualitative resemblance to the conduction solution (2.15); and the same is true of the measurements of Mei, Liu & Carter (1972). There are many possible reasons for this. Elapsed time and distance downstream from the wavemaker may have been insufficient for the complete diffusion of the viscous boundary layers; wave amplitudes may sometimes have been too large for meaningful comparison with weakly nonlinear theory; the use of Teepol profoundly alters the surface boundary conditions (see § 4) for short waves; the longer waves are affected by the air boundary layer (Dore 1978*a*, *b*); unidirectional drift profiles are likely to be unstable to spanwise-periodic disturbances (see § 7); and side walls may exert a direct influence on the drift profiles. The various attempts to compare theory with these experiments – e.g. by Liu & Davis (1977) (who in any case erroneously include an exponential decay factor twice over in their curves for $t \neq 0$) and Grimshaw (1981) – should therefore not be taken too seriously. A truly definitive experiment on drift profiles is still lacking, so long after Stokes's pioneering paper.

4. Periodic waves with surface contamination

The use of curvilinear co-ordinates, which fit the distorted free surface, is preferable to a Cartesian formulation. Essentially, the Cartesian formulation is justifiable only for wave amplitudes small compared with the viscous-boundary-layer thickness $(\sigma/2\nu)^{-\frac{1}{2}} = \beta^{-1}$; a severe restriction which may be unnecessary if suitable curvilinear co-ordinates are used. However, as a first approach to the present problem, a Cartesian analysis was developed because of its greater simplicity: the results so obtained agreed with the subsequent curvilinear analysis. For brevity, only the latter analysis is described here. We assume that the viscosity is sufficiently small that $|\sigma|/k^2\nu \gg 1$, and that the wave is maintained at constant amplitude by a suitable distribution of periodic normal stresses at the free surface. The bottom boundary conditions are $u = w = 0$ at $z = -d$.

In the uncontaminated case, the free-surface boundary condition is that the linearized tangential stress there be zero. However, with surface contamination, this condition is no longer applicable. Instead, we here adopt the 'inextensible' model of the surface film which is known to give acceptable results under appropriate circumstances (see § 2.3 above).

Within the surface boundary layer we adopt the orthogonal curvilinear co-ordinates

$$\left.\begin{aligned} \xi &= x' - \frac{a\cosh[k(z+d)]\sin kx'}{\sinh kd}, \\ \eta &= z - \frac{a\sinh[k(z+d)]\cos kx'}{\sinh kd}, \end{aligned}\right\} \tag{4.1}$$

where $x' = x-(\sigma/k)t$. This reference frame is fixed relative to the wave and the *inviscid* irrotational solution equivalent to (2.1) is given by the stream function

$\Psi_0 = -(\sigma/k)\,\eta$. More generally, we write $\Psi = -(\sigma/k)\,\eta + \Psi_1(\xi,\eta)$, the velocity components (v_ξ, v_η) in the directions of increasing ξ and η being

$$v_\xi = J^{\frac{1}{2}}\partial\Psi/\partial\eta, \quad v_\eta = -J^{\frac{1}{2}}\partial\Psi/\partial\xi, \tag{4.2}$$

where

$$J \equiv \left\{1 - \frac{2ak\cosh\left[k(z+d)\right]\cos kx'}{\sinh kd} + \frac{a^2k^2[\cosh^2 k(z+d) - \sin^2 kx]}{\sinh^2 kd}\right\}$$

is the Jacobian $\partial(\xi,\eta)/\partial(x',z)$.

The condition that the surface is inextensible is

$$a\sigma\coth kd\cos k\xi + (\partial\Psi_1/\partial\eta)_{\eta=0} = 0 \tag{4.3}$$

in the linear approximation. Within the viscous boundary layer near $\eta = 0$, the spanwise vorticity ω has the form

$$\omega(\xi,\eta) = \mathrm{Re}\{\mathscr{A}\exp\left[ik\xi + \beta\eta(1-i)\right]\} \tag{4.4}$$

in the linear approximation, where $\mathscr{A}$ is a constant and

$$\omega = J(\partial^2\Psi_1/\partial\xi^2 + \partial^2\Psi_1/\partial\eta^2) \simeq \partial^2\Psi_1/\partial\eta^2$$

(cf. Phillips 1977, p. 47). This may be integrated to yield the linearized stream function

$$\Psi_1(\xi,\eta) = \mathrm{Re}\{a\sigma(2\beta)^{-1}\coth kd(1+i)\,e^{ik\xi}[1 - e^{\beta\eta(1-i)}]\} \tag{4.5}$$

on imposing the boundary conditions $v_\eta(\xi,0) = 0$, (4.3) and $\partial^2\Psi_1/\partial\eta^2 \to 0$ as $\beta\eta \to -\infty$. We note that, as $\beta\eta \to -\infty$, Ψ_1 remains non-zero; this represents the modification to the potential flow induced by the surface boundary layer.

The mean vorticity equation is

$$-\nu\,\overline{\nabla\times(\nabla\times\boldsymbol{\omega})} = \overline{\nabla\times(\boldsymbol{\omega}\times\mathbf{u})},$$

which, in the vicinity of the surface $\eta = 0$, reduces to

$$\nu[\partial^4\Psi_2/\partial\eta^4 - 2a^2\sigma k\beta^3\coth^2 kd\,e^{\beta\eta}(\cos\beta\eta + \sin\beta\eta)]$$
$$= -a^2\sigma^2 k\beta\coth^2 kd\,e^{\beta\eta}(\cos\beta\eta + \sin\beta\eta - e^{\beta\eta}), \tag{4.6}$$

correct to $O(a^2)$, on using (4.5). Here, $\Psi_2(\xi,\eta)$ denotes the stream function associated with second-order mean motion. Since $\beta^2 = \sigma/2\nu$, this reduces to

$$\partial^4\Psi_2/\partial\eta^4 = 2a^2\sigma k\beta^3\coth^2 kd\,e^{2\beta\eta}. \tag{4.7}$$

Integration gives

$$\partial^2\Psi_2/\partial\eta^2 = \tfrac{1}{2}a^2\sigma k\beta\coth^2 kd\,e^{2\beta\eta} + F(\xi), \tag{4.8}$$

if it is assumed that $\partial^3\Psi_2/\partial\eta^3 \to 0$ as $\beta\eta \to \infty$ (which neglects the influence of any mean pressure gradient $\overline{dp/dx}$: a justifiable assumption).

The mean tangential stress at the surface $\eta = 0$ is

$$\bar{\tau}_{\xi\eta} = \rho\nu\overline{\left[\frac{\partial}{\partial\eta}\left(J\frac{\partial\Psi}{\partial\eta}\right) - \frac{\partial}{\partial\xi}\left(J\frac{\partial\Psi}{\partial\xi}\right)\right]}.$$

If this is taken to be zero, as previously, then

$$\partial^2\Psi_2/\partial\eta^2 = a^2\sigma k\beta\coth^2 kd \quad (\eta = 0). \tag{4.9}$$

It follows that

$$F(\xi) = \tfrac{1}{2}a^2\sigma k\beta \coth^2 kd,$$

and hence that

$$\partial^2\Psi_2/\partial\eta^2 \rightarrow \tfrac{1}{2}a^2\sigma k\beta \coth^2 kd \quad (\beta\eta \rightarrow -\infty) \tag{4.10}$$

just outside the surface boundary layer.

In terms of the Cartesian co-ordinates x', z the mean Eulerian velocity gradient near the surface is

$$\frac{\partial \bar{u}_e}{\partial z} = \frac{\partial^2\Psi_2}{\partial z^2} = \left(\frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} + \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}\right)^2 \Psi_2, \tag{4.11}$$

which may be evaluated just outside the surface boundary layer ($\beta\eta \rightarrow -\infty$, $z \simeq 0$) as

$$\left(\frac{d\bar{u}_e}{dz}\right)_{z=0} = \left(\frac{\partial^2\Psi_2}{\partial\eta^2}\right)_{\beta\eta\rightarrow-\infty} + O(a^2k^2\sigma) = \tfrac{1}{2}a^2\sigma k\beta \coth^2 kd[1+O(k\beta^{-1})]. \tag{4.12}$$

The mean Eulerian velocity $\bar{u}_e(z)$ satisfies (2.3) in the interior, and the bottom boundary condition (2.7) is essentially unaltered by the presence of surface contamination. It is easily verified, *a posteriori*, that any influence on the free-surface boundary condition from a mean pressure gradient $\overline{dp/dx}$ required to remain zero mass flux (which was ignored in deriving (4.12)) is negligible compared with the leading-order term of (4.12).

Condition (4.12) replaces (2.9) and shows that the mean velocity gradient below an inextensible surface is much greater than that below a clean one. Dore (1970, 1978*a*, *b*) found a similar result for interfacial waves on a clean interface (see § 7) and pointed out that the air boundary layer greatly affects surface gravity waves with wavelengths of a metre or more. Since the inextensible free-surface model is most appropriate for *short* gravity and capillary-gravity waves, it is likely that, with surface contamination, the boundary condition (2.9) may *never* be a good approximation for any wavelength!

The key to understanding these surprising results lies in the linear damping rate of the waves. In the present model, the waves are of constant amplitude, being maintained against viscosity by periodic normal stresses. As described in § 2, these stresses supply momentum to the waves at a rate which equals the rate at which they lose momentum by viscous action. Since momentum is also lost in exerting a net force on the rigid lower boundary, the rate of momentum loss to the mean flow is usually less than the total loss. The difference between the total loss and the loss to the lower boundary must equal the viscous stress of the Eulerian mean flow. For deep-water waves and for waves on contaminated surfaces, the loss to the lower boundary is small in comparison to the total loss.

More precisely, the total mean stress below the level of wave troughs equals the mean horizontal force F per unit area owing to the periodic normal stresses at the surface. Since the mean Reynolds stress $-\rho(\overline{uw})_\infty$ is constrained by the bottom boundary layer to be as in (2.5) – which vanishes for deep water – and, since the surface film is assumed to exert no mean horizontal stress, it follows that

$$\rho\nu(d\bar{u}_e/dz)_{z=0} = F + \rho(\overline{uw})_\infty.$$

But the rate of working of F is $(\sigma/k)\,F$ and this equals the rate at which wave energy E per unit area would decay in the absence of forcing. Since

$$E = (2k)^{-1}\,\rho\sigma^2 a^2 \coth kd$$

(Phillips 1977, chap. 3), we have

$$F = \rho\sigma_{\mathrm{i}}\,\sigma a^2 \coth kd,$$

$$\nu(d\bar{u}_{\mathrm{e}}/dz)_{z=0} = \sigma a^2 \coth kd(\sigma_{\mathrm{i}} - \tilde{\sigma}_{\mathrm{i}}) \tag{4.13}$$

where $\tilde{\sigma}_{\mathrm{i}}$ denotes the temporal decay rate (2.16) due to bottom friction alone.

This result agrees with (2.9) for clean surfaces where σ_{i} is given by (2.17); and also with (4.12) for inextensible films, where σ_{i} is given by (2.23) to leading order and $\tilde{\sigma}_{\mathrm{i}}/\sigma_{\mathrm{i}}$ is negligibly small. Equation (4.13) also gives the correct generalization for contaminated but *extensible* elastic films on substituting the appropriate value for σ_{i} (see § 2). For unforced waves, similar arguments apply and (4.13) is unaltered apart from the appearance of a decay factor $\exp(-2\sigma_{\mathrm{i}}t)$ on the right-hand side.

The Stokes drift (2.2) is unchanged by the presence of contamination, as is the structure of the bottom boundary layer which yields the boundary condition (2.7) for the interior mean flow. However, the latter boundary condition requires some slight qualification in view of the large mean-flow gradients implied by the condition (4.12) for contaminated surfaces. With an inextensible surface, (4.12) and (2.3) yield

$$d\bar{u}_{\mathrm{e}}/dz = (\nu\rho)^{-1}\,(\overline{dp/dx})\,d + \tfrac{1}{2}(\sigma/2\nu)^{\frac{1}{2}}\,\sigma a^2 k \coth^2 kd$$

if $\bar{u}_{\mathrm{e}}$ is a steady flow. The bottom-boundary-layer solution then consists of (2.6) plus a term

$$[(\nu\rho)^{-1}\,(\overline{dp/dx})\,d + \tfrac{1}{2}(\sigma/2\nu)^{\frac{1}{2}}\,\sigma a^2 k \coth^2 kd]\,(z+d),$$

which is no longer negligible at distances $z+d$ of order β^{-1} from the bottom.

With zero pressure gradient, the steady-state Eulerian mean flow in the interior is

$$\bar{u}_{\mathrm{e}}(z) = \bar{u}_{\mathrm{e}}(-d) + (d\bar{u}_{\mathrm{e}}/dz)_{z=0}\,(z+d),$$

where $\bar{u}_{\mathrm{e}}(-d)$ is given by (2.7) and $(d\bar{u}_{\mathrm{e}}/dz)_{z=0}$ by (4.12), for an inextensible surface. Since the latter is now so large, the bottom-velocity contribution $\bar{u}_{\mathrm{e}}(-d)$ is small over most of the interior region and

$$\bar{u}_{\mathrm{e}}(z) = \tfrac{1}{2}(\sigma/2\nu)^{\frac{1}{2}}\,\sigma a^2 k \coth^2 kd(z+d) \tag{4.14}$$

to leading order. With zero mass flux, the corresponding velocity distribution is

$$\bar{u}_{\mathrm{e}}(z) = \tfrac{1}{2}(\sigma/2\nu)^{\frac{1}{2}}\,\sigma a^2 k \coth^2 kd[z + d + \tfrac{3}{4}d^{-1}(z^2 - d^2)] \tag{4.15}$$

to the same order of approximation. This yields a theoretical velocity $\bar{u}_{\mathrm{e}}(0)$ near the surface which is much greater than the bottom velocity (2.7). Also (4.15) far exceeds the Stokes drift (2.2) for most values of z. Note that the requirement $a\beta \ll 1$, which would have been necessary for the validity of a Cartesian analysis of the surface boundary layer, is now the condition that the mean velocity (4.15) remains small compared with the wave orbital velocity. Since the latter requirement is essential for any weakly nonlinear theory, it seems that a Cartesian analysis would not have imposed *unnecessarily* stringent assumptions in the present case, in contrast with the analysis for a clean surface. This result was unexpected.

5. Temporally decaying waves

With a clean surface, the mean Eulerian flow associated with temporally decaying waves satisfies (2.20) and the boundary conditions (2.18) and (2.19). A particular solution is (2.22) essentially as found by Liu & Davis (1977). Here we consider the initial-value problem defined by (2.18)–(2.20) and the initial condition $\overline{u}_e(z,t) = 0$ at time $t = 0$ corresponding to the onset of wave motion. We may represent the solution as

$$\overline{u}_e(z,t) = \overline{u}_p(z,t) + \overline{u}_c(z,t)$$

where $\overline{u}_p(z,t)$ denotes the particular solution (2.22) and $\overline{u}_c(z,t)$ satisfies

$$\left.\begin{aligned} &\partial\overline{u}_c/\partial t - \nu(\partial^2\overline{u}_c/\partial z^2) = 0,\\ &\overline{u}_c(-d,t) = 0, \quad \partial\overline{u}_c(0,t)/\partial z = 0,\\ &\overline{u}_c(z,0) = -\overline{u}_p(z,0), \end{aligned}\right\} \tag{5.1}$$

provided no mean pressure gradient other than that of (2.22) is present. This has the solution

$$\overline{u}_c(z,t) = \sum_{n=1}^{\infty} A_n \cos\lambda_n z \exp(-\nu\lambda_n^2 t), \tag{5.2}$$

where

$$\lambda_n = -(\pi/d)(n-\tfrac{1}{2}) \quad (n = 1,2,3,\ldots),$$

and the coefficients A_n satisfy

$$\sum_{n=1}^{\infty} A_n \cos\lambda_n z = -(P/2\sigma_i\rho) - (\delta^{-1}\sin\delta z)(2\sigma a^2k^2\coth kd) + \frac{\cos\delta z}{\cos\delta d}\left[\frac{P}{2\sigma_i\rho} - 2\delta^{-1}\sigma a^2k^2\coth kd\sin\delta d - \frac{3\sigma a^2 k}{4\sinh^2 kd}\right]. \tag{5.3}$$

If, in fact, the net mass flux is zero at each instant, it is necessary to impose an additional pressure gradient $\overline{\partial p_c/\partial x}$ in (5.1) which is a function of time alone. However, we here assume that $\overline{\partial p/\partial x} = 0$ and hence that P in (5.3) is also zero. This allows non-zero mass flux, as is appropriate to an unbounded, spatially periodic model.

It follows that

$$\begin{aligned} A_N = &-\frac{4\sigma a^2k^2\coth kd}{\delta d\cos\delta d}\\ &\times\int_{-d}^{0}\sin[\delta(z+d)]\cos\lambda_N z\,dz - \frac{3\sigma a^2k}{2d\cos\delta d\sinh^2 kd}\int_{-d}^{0}\cos\delta z\cos\lambda_N z\,dz\\ = &\frac{-\sigma a^2 k}{d(\delta^2-\lambda_N^2)}\left[\frac{4k\lambda_N\coth kd}{\delta} + \frac{3(-1)^N\delta}{2\sinh^2 kd}\right] \quad (N = 1,2,3,\ldots). \end{aligned} \tag{5.4}$$

However, this solution breaks down at values of d such that $\lambda_N^2 = \delta^2$ and separate treatment of such cases is necessary. These values of d are just the singular depths of Liu & Davis's (1977) particular integral (2.22).

At large times t, the dominant contribution to $\overline{u}_c$ derives from the term in A_1, except possibly very close to the boundaries $z = 0, -d$. Then,

$$\overline{u}_c(z,t) \sim A_1\cos\left(\frac{\pi z}{2d}\right)\exp\left(\frac{-\pi^2\nu t}{4d^2}\right) \quad (t\to\infty). \tag{5.5}$$

This term decays more or less rapidly than the particular solution (2.22) accordingly as

$$\pi^2\nu/4d^2 \gtrless 2\sigma_i. \tag{5.6}$$

For deep-water waves, $\sigma_i = 2\nu k^2$, and this condition becomes $kd \lessgtr \frac{1}{4}\pi$. Since kd certainly exceeds $\frac{1}{4}\pi$ for deep-water waves, the dominant term as $t \rightarrow \infty$ is therefore (5.5) and *not* (2.22). For shallow-water waves, with σ_i given by (2.16), condition (5.6) is

$$\pi^2/8d^2 \gtrless (\sigma/2\nu)^{\frac{1}{2}}\, k \operatorname{cosech} 2kd$$

(where the right-hand side must be large compared with k^2 in order that result (2.16) applies). Again, (5.5) rather than (2.22) is certain to dominate as $t \rightarrow \infty$ in most cases of interest.

The singularities of the particular solution (2.22) arise at values of $2\sigma_i$ which coincide with one or other of the decay rates $\nu\lambda_M^2$ of the modes of (5.2). The solution to the initial-value problem is then modified as follows. Suppose that $2\sigma_i = \nu\lambda_M^2$ for a particular integer M: then the A_N $(N \neq M)$ are given by (5.4) as before, where δ now equals $-\lambda_M$. Also, taking the sum of the particular solution (2.22) and the term in A_M and considering the limit $2\sigma_i \rightarrow \nu\lambda_M^2$ leads to a *non-singular* particular solution of (2.18)–(2.20) incorporating terms of the form

$$\exp(-\nu\lambda_M^2 t)\,[a_1 \cos\lambda_M z + a_2 \sin\lambda_M z + a_3(\nu\lambda_M^2 t \cos\lambda_M z + \lambda_M z \sin\lambda_M z)],$$

where a_1, a_2, a_3 are appropriate finite constants. Such terms have no singularity as $z \rightarrow \infty$ since the constant a_3 is proportional to d^{-1} as the depth $d \rightarrow \infty$. The singularities of (2.22) have therefore been resolved and shown to be of no physical significance. A similar conclusion was reached independently by Grimshaw (1981 and private communication) by using Laplace transforms to solve the initial-value problem.

The above treatment of time-dependent waves may readily be extended to decaying waves with a contaminated surface. Since the decay rate is still quite small in this situation, the mean flow within the viscous boundary layers may be regarded as quasi-steady and the boundary conditions just outside the viscous regions are then (2.7) and (4.12) or its generalization (4.13). Solutions equivalent to (2.22) and (5.2) are easily obtained and again exhibit no singular behaviour.

6. Spatially decaying waves

With spatial decay, the mean drift velocity cannot be strictly unidirectional since any x-variations in $\bar{u}_e$ must be associated with a non-zero vertical velocity $\bar{w}_e$. On writing

$$\bar{u}_e(x,z,t) = \partial\Psi_2/\partial z, \quad \bar{w}_e(x,z,t) = -\partial\Psi_2/\partial x,$$

where $\Psi_2(x,z,t)$ denotes the stream function of the mean velocities, the mean-vorticity equation becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial\Psi_2}{\partial z}\frac{\partial}{\partial x} - \frac{\partial\Psi_2}{\partial x}\frac{\partial}{\partial z}\right)\nabla^2\Psi_2 - \nu\nabla^2\nabla^2\Psi_2 = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2}\right)\overline{uw} + \frac{\partial^2}{\partial x\,\partial z}(\overline{w^2} - \overline{u^2}), \tag{6.1}$$

where the right-hand side derives from the mean Reynolds stresses associated with the wave motion and the overbar now denotes a time-average.

In the interior of the fluid, beyond the viscous boundary layers, the wave motion is described, as a first approximation, by the irrotational solution of the form

$$u = \mathrm{Re}\{ik\hat{\phi}\exp[i(kx-\sigma t)]\}, \quad w = \mathrm{Re}\{\hat{\phi}'\exp[i(kx-\sigma t)]\},$$
$$\hat{\phi} = A\,e^{-kz}+B\,e^{kz},$$

where the prime (′) denotes d/dz. Such a solution remains appropriate for waves subject to slow spatial, as well as temporal, attenuation. We suppose that the frequency σ is real, but the wavenumber $k = k_r + ik_i$ now has a small non-zero imaginary part, related to the temporal decay rate discussed previously by

$$k_i = \sigma_i c_g^{-1}.$$

Here, $c_g = d\sigma/dk$ is the group velocity of the waves, which may be regarded as real to sufficient accuracy.

In such cases, the right-hand side of (6.1) is identically zero. On assuming, also, that the convective terms on the left-hand side of (6.1) are negligible (not necessarily a good assumption, but certainly true for small enough wave amplitudes, and large-enough distances from the wavemaker), (6.1) reduces to the biharmonic equation. A steady-state solution exists with the form $\Psi_2 = \Phi_2\exp(-2k_i x)$, where

$$(d^2/dz^2+4k_i^2)^2\,\Phi_2 = 0. \tag{6.2}$$

Equation (6.2) has the general solution

$$\Phi_2 = (A_1+C_1 z)\sin(2k_i z)+(B_1+D_1 z)\cos 2k_i z. \tag{6.3}$$

It is reasonable to suppose that, for realistically small decay rates k_i, the boundary conditions at $z = 0$ and $-d$ are essentially unaltered from those for unattenuated waves: this was verified *a posteriori*, in agreement with the independent analysis of Grimshaw (1981). Accordingly, we set

$$\left.\begin{aligned}\Phi_2(-d) = 0, \quad \Phi_2'(-d) = \frac{3\sigma a^2 k_r}{4\sinh^2 k_r d} \equiv \mathscr{P},\\ \Phi_2(0) = \mathscr{Q}, \quad \Phi_2''(0) = 2\sigma a^2 k_r^2\coth k_r d \equiv \mathscr{R},\end{aligned}\right\} \tag{6.4}$$

where the prime denotes d/dz and $\mathscr{Q}\exp(-2k_i x)$ is the net horizontal Eulerian mass flux at each station x. For closed channels, $\mathscr{Q}$ must be chosen to cancel the downstream mass flux associated with the Stokes drift. These boundary conditions yield the appropriate values of A_1, B_1, C_1 and D_1 in (6.3).

When $k_i d$ is small – which is typically so – Φ_2 is given approximately by

$$d^4\Phi_2/dz^4 = 0,$$

which yields the solution

$$\Phi_2 \simeq \frac{1}{4}\left(1+\frac{z}{d}\right)^2\left[\mathscr{Q}\left(2-\frac{z}{d}\right)+\mathscr{R}zd\right]+\tfrac{1}{2}\mathscr{P}z\left(\frac{z^2}{d^2}-1\right)+O(\sigma a^2 k_i^2 d^2), \tag{6.5}$$

with mean pressure gradient $\overline{dp/dx} = 3(\mathscr{P}d-\tfrac{1}{2}\mathscr{Q})\,d^{-3}\nu\rho$.

This approximation yields Longuet-Higgins' conduction solution for $\bar{u}_e = \partial\Psi_2/\partial z$, modified by the attenuation factor $\exp(-2k_i x)$. Only when the downstream 'attenuation length' k_i^{-1} becomes comparable with (or less than) the liquid depth d does (6.3) give significant departures from this solution.

The range of validity of the approximation (6.5) requires clarification. This solution is valid at times and distances downstream from the wavenumber sufficiently large that viscous diffusion extends throughout the depth d and convection effects are negligible. The solution therefore complements the convection-dominated steady boundary-layer solution of Dore (1977) which is valid nearer the wavemaker. The time-dependent analysis of Grimshaw (1981) treats, among other cases, that of the developing solution downstream of a wavemaker started at time $t = 0$: this also leads to the conduction solution as $t \to \infty$, but Grimshaw's more general analysis is inevitably more complicated than the present account.

7. Instability of the drift profile

The various drift-velocity profiles described above consist of two parts: the Eulerian component $\bar{u}_e$ and the Stokes drift $\bar{u}_s$. A satisfactory stability analysis of such drift velocities must account for the separate roles of these components. Fortunately, such an analysis already exists in the work of Craik & Leibovich (1976), Craik (1977) and more recent papers, particularly Leibovich & Paolucci (1980, 1981). This work was developed to describe the onset of Langmuir circulations in bodies of water subject to a wind stress, the essential dynamical mechanism being a coupling between the wind-driven Eulerian current and the Stokes drift of the surface-wave field. The governing equations were derived by Craik & Leibovich (1976) (see also Leibovich (1980) for an alternative derivation) who calculated the secondary spanwise-periodic current system associated with a pair of monochromatic wave trains propagating obliquely to the wind direction. Using the same governing equations, Craik (1977) subsequently showed that initially unidirectional drift profiles are normally unstable to spanwise-periodic disturbances. Extensive computer solutions of the developing flow and further clarification of the stability properties are given by Leibovich & Paolucci (1980, 1981), who also investigate the influence of a stable density stratification. In addition to the various reported observations of Langmuir circulations in lakes and ocean, this phenomenon has been produced in the laboratory by Faller & Caponi (1978) and Faller (1978).

In all the above-mentioned work, the mean Eulerian current was regarded as deriving from an applied wind stress, whereas the present paper concerns the Eulerian contribution driven by viscous boundary layers in the *absence* of wind. Despite this distinction, all of the above work is based on scaling assumptions identical with those used here and the results may be carried over directly to the present context. Inclusion of a wind-driven component in the Eulerian drift velocity would constitute a straightforward extension of the present work.

Mathematically, the linear instability of spanwise-periodic disturbances is akin to that of thermal-convection rolls; but the physical mechanism is entirely different, the essential process involving tilting of vortex lines by the Stokes drift gradient (see Craik 1977). Whereas Craik (1977) invoked the analogy with thermal convection to establish an approximate stability criterion, the recent work of Leibovich & Paolucci (1981) has now derived this precisely. They consider a monochromatic train of gravity waves in deep water, in which the mean velocity profile develops from rest in response to a constant surface stress τ_0 imposed at all times $t \geqslant 0$. As well as the case of constant density, they treat stable uniform density gradients.

Although Leibovich & Paolucci (1981) regarded the stress τ_0 as due to a wind initiated at $t = 0$, it may just as readily be attributed to the free-surface boundary condition, e.g. (2.9), induced by viscous action on the waves. More precisely, if we suppose that a uniform wave field is rapidly set up at $t = 0$ and that the waves are subsequently maintained at constant amplitude by suitable periodic normal stresses at the surface, the analysis of Leibovich & Paolucci (1981) remains valid on replacing their 'wind stress' $\tau_0 \equiv \rho u_*^2$ by $\tau_0 \equiv \rho\nu\bar{u}_e'(0)$ where $\bar{u}_e'(0)$ is the velocity gradient determined by the free-surface boundary condition. For deep water of constant density and with a clean surface and no air boundary layer, Longuet-Higgins' result (2.9) yields

$$\tau_0 = \rho\nu 2\sigma(ak)^2. \tag{7.1}$$

The alternative values with contamination or air boundary layer are usually much greater than this.

Now, Leibovich & Paolucci found good agreement between a global energy-stability criterion, valid for nonlinear disturbances, and the results of a linear-stability analysis. For unstratified flow, they found global stability whenever a characteristic 'inverse Langmuir number'

$$L^{-1} \equiv \frac{au_*}{k\nu}\left(\frac{\sigma}{\nu}\right)^{\frac{1}{2}}$$

is less than 1·46; and linear instability whenever L^{-1} exceeds 1·52. The critical wave-number of linearized spanwise-periodic disturbances is $0{\cdot}32k$, where k is the wave-number of the gravity waves; and the optimum global stability limit was also obtained at this value.

Here, ν denotes a constant eddy viscosity or, under laminar conditions, the actual kinematic viscosity of water. On replacing u_* by $(2\nu\sigma)^{\frac{1}{2}}ak$, in line with (7.1), we obtain the global-stability criterion

$$a^2\sigma\nu^{-1} < 1{\cdot}03, \tag{7.2a}$$

and the linear-instability criterion

$$a^2\sigma\nu^{-1} > 1{\cdot}07. \tag{7.2b}$$

The latter condition is normally well satisfied, except for waves of extremely small amplitude. For instance, with frequency σ appropriate to wavelengths of 1 m and with kinematic viscosity $\nu = 10^{-2}$ cm^2 s^{-1}, it is satisfied for all amplitudes a greater than 0·037 cm! The corresponding condition with surface contamination or air boundary layer is even more easily satisfied.

We may conclude with certainty that the unidirectional drift currents predicted by the theories described above will be unstable to spanwise-periodic disturbances, provided the depth or spanwise dimension of the flow is not unduly confined. It is possible, too, that such drift currents may be unstable to other modes of disturbance which vary in the downstream direction; but such modes are outside the scope of the present work. In any case, the existence of instability has been demonstrated and a lack of good agreement between two-dimensional theory and experiment – except near the bottom wall and free surface – seems likely to remain.

8. Interfacial waves: a postscript

Drift-velocity profiles for progressive interfacial waves in a two-fluid system have been calculated by Dore (1970, 1978*b*) and Dore & Al-Zanaidi (1979). They find induced Eulerian currents much larger – by a factor $O[(\sigma/k^2\nu)^{\frac{1}{2}}]$ – than those for a single fluid with a clean surface, provided the density difference remains large. Even for small density differences, as with a salt-water–fresh-water interface, an internal wave produces substantial mass flux. This result, at first sight surprising, is entirely consistent with the analysis of § 4 above for a contaminated inextensible surface. In both situations, the viscous boundary layers near the surface are much intensified and the appropriate drift velocity gradient $d\bar{u}_e/dz$ outside these layers is $O(\sigma a^2 k\beta)$ as in (4.13): this is in line with the increased linear damping rate of waves, as discussed in § 4 above.

The stability of such interfacial drift profiles is of particular interest, in view of its relevance to the dynamics of the oceanic thermocline. A necessary requirement for instability to spanwise-periodic disturbances is that the gradients of Eulerian and Stokes components of the drift velocity are of the same sign. In Dore's solutions, this is so for a considerable distance, say d_1, on either side of the interface. An 'effective Rayleigh number' is

$$Ra = d_1^4 \nu^{-2}(d\bar{u}_e/dz)(d\bar{u}_s/dz)$$

on invoking the analogy with thermal convection (e.g. Craik 1977), where ν is a measure of the respective viscosities (assumed comparable) and the gradients of $\bar{u}_e$ and $\bar{u}_s$ may be evaluated just beyond the interface. On supposing, also, that kd_1 is $O(1)$ or more, Ra is found to be of order

$$Ra \sim O\{(ad_1)^4 k^{\frac{7}{4}}(g\Delta\rho/\rho)^{\frac{5}{4}} \nu^{-\frac{5}{2}}\},$$

where $\Delta\rho$ denotes the density difference across the interface. This is likely to be a large number, indicative of instability, for circumstances typical of the laboratory and of lowest-mode internal waves on the ocean thermocline.

As with wind-driven Langmuir circulations, but now in the *absence* of wind, instability of the surface-wave drift current (as discussed in § 7) will provide an effective mixing mechanism near the ocean surface. Not only do the longitudinal eddies themselves act to mix the fluid. Near down-welling regions, the local downstream velocity profile may develop rather strong shear at considerable depths (cf. the computed profiles of Leibovich & Paolucci 1980) and such profiles may themselves be unstable, forming intermittent internal 'billows' which further enhance mixing. Instability of the latter sort has been extensively studied by Thorpe (see for example Thorpe (1977), who gives other references).

The drift velocity associated with lowest-mode internal waves is also likely to be unstable to spanwise-periodic disturbances; but it is not entirely clear how this would affect the thermocline structure. With long waves on a thermocline, a spanwise-periodic instability of the drift-current profile would produce a separate array of roll cells on either side of the density 'jump'. In the author's opinion, the associated vertical mixing would tend to cause an increase, rather than a reduction, in the density gradient within the thermocline. If this is so, the presence of small-amplitude internal waves may act to maintain a sharp thermocline against erosion by molecular diffusion and other small-scale mixing motions. Experimental and observational verification of the instability of drift currents, and further measurements of actual drift-current profiles, are desirable.

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